NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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ANALYTICAL METHOD FOR DETERMINING TRANSMISSION AND ABSORPTION OF TIME-DEPENDENT RADIATION THROUGH THICK ABSORBERS

I - RADIOACTIVITY OF ABSORBER, TIME-DEPENDENT

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SUMMARY

A theoretical analysis of absorbers whose radioactivity is time-dependent is presented. Matrix methods are employed as a tool in the analytical determination of the intenstiy of radioactivity and the amount of heat generated in any portion of a thick absorber. The method is applied to the following three cases: (1) plane source of monochromatic radiation of a single type at normal incidence to a plane absorber, (2) plane source of polychromatic radiation of a single type at normal incidence to a plane absorber, and (3) plane source of polychromatic radiation of several types at normal incidence to a plane absorber. Illustrative examples are included.

Experimental data on the absorption of gamma rays and neutrons and on the rates of conversion of these radiations to thermal energy are required for utilization of this method.

INTRODUCTION

A mathematical analysis of the interaction of nuclear radiations with matter requires simultaneous consideration of a large number of factors. Fortunately, many of these factors obey the same mathematical laws. The compact notation inherent in matrix methods makes them ideal for problems of this nature.

The matrix method was first used to calculate the transmission and reflection of thermal radiations from the surfaces of a number of thin foils (reference 1). A similar method has been applied to nuclear radiations and developed for solutions of cases of polychromatic incident radiation, neutron and gamma-ray combinations, radiation not normal to the face of the absorber, and time-independent radioactivity in the shield itself (reference 2).

Inasmuch as a great many artificially produced radioactive substances have very short half-lives, that is, the radioactivity of the substances decays rapidly, the extension of the method to include time-dependent cases was considered useful in widening its applicability. Thus, an extension of the method was developed at the NACA Lewis laboratory to include the following time-dependent cases:

(1) The incident radiation is monochromatic. The radiation is normal to the absorber. Part of the radiation absorbed at each station of the plane absorber is transformed to heat and part to radioactivity in the absorber. The absorber emits radiation of the same type and wavelength as the incident radiation. The radioactivity produced in each station of the absorber is entirely emitted.

(2) An extension of case (1) to include polychromatic radiation

(3) An extension of case (2) to include mixtures of gamma and neutron radiation

As used herein, the expression "type of radiation" refers to the fundamental particles emitted by radioactive elements. Gamma radiation is one type of radiation; neutron radiation is another type.

Case (1), which assumes no degradation of energy, is presented for the sole purpose of illustrating the use of the method with a minimum of cumbersome notations and computations.

Experimentally verifiable data on all the constants used must be supplied for proper utilization of the method. Such constants include absorption coefficients, reflection coefficients, transmission coefficients, rates of conversion of energy of nuclear radiations to thermal energy, and radioactive decay constants. For gamma rays, most of the required absorption data are already available or can be computed; but for neutrons, the amount of available data, although large, is still much smaller than is to be desired.

ANALYSIS

This analysis may be conveniently divided into three separate cases. Each of these cases, together with illustrative examples, are subsequently described. All symbols are defined in appendix A.

Case I

The assumptions used in case I are:

(1) Stations of absorber are parallel plane surfaces.

(2) Incident radiation is monochromatic.

(3) Radiation is normal to absorbers.

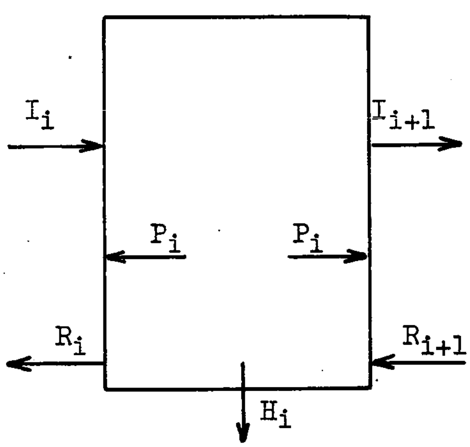
(4) No degradation of energy occurs.

(5) Part of energy of radiation absorbed is transformed to energy of radioactivity and part to thermal energy.

(6) Absorber emits radiation of same type and energy as incident radiation.

(7) Radioactivity produced in each station of absorber is entirely emitted.

Method. - Diagram (a) is included to clarify the discussion in the following paragraph.



(a)

The absorber may be considered as consisting of n stations normal to the path of the radiation. The i^{th} station of the absorber is denoted by S_i. Radiation I_i from the preceding station S_{i-1} is incident upon one surface of S_i, while radiation R_{i+1} from the subsequent station S_{i+1} impinges upon the opposite surface of S_i. From S_i emerges radiation I_{i+1}, which is incident upon S_{i+1}, and radiation R_i, which is incident upon S_{i-1}.

In the treatment presented herein, the following quantities are assumed to be known time-independent constants:

I_1	radiation power incident upon absorber from source
R_{n+1}	radiation power incident upon absorber from surroundings outside absorber
$(P_i)_{\tau = \tau_0}$	one-half of initial power of radioactivity in each station of absorber
$(H_i)_{\tau = \tau_0}$	initial thermal power being emitted from each station of absorber
t_i	power-transmission coefficient
r_i	power back-scattering coefficient
μ_i	rate of conversion of energy of radiation absorbed to energy of radioactivity
λ_i	radioactive decay constant
h_i	rate of conversion of energy of nuclear radiation absorbed to thermal energy for each station of absorber

The problem is to find I_i and R_i in terms of known quantities. The following procedure is used to solve the problem:

All I_i and R_i are expressed in terms of I_1, R_{n+1}, t_i, r_i, and P_i. These expressions are then substituted in the differential equations for P_i and the resulting set of n simultaneous linear differential equations of the first order are solved for P_i. This procedure will yield P_i in terms of known quantities; when these values of P_i are substituted in the expressions for I_i and R_i, the original problem will be solved. If these expressions for I_i and R_i are substituted into the equation expressing the conversion of radioactive energy to thermal energy, H_i can be found.

For the i^{th} station of the absorber S_i (diagram (a)), the following relations hold:

$$I_{i+1} = t_iI_i + r_i R_{i+1} + P_i \tag{1}$$

$$R_i = r_iI_i + t_i R_{i+1} + P_i \tag{2}$$

$$H_i = h_i (I_i + R_{i+1}) \tag{3}$$

$$\frac{dP_i}{d\tau} = \mu_i (I_i + R_{i+1}) - \lambda_iP_i \tag{4}$$

The derivation of equation (4) is given in appendix B.

Equation (4) is the basic feature of the treatment of time-dependency in this paper. The absence of additional terms in this equation (and also in equation (3)) is due to the simplifying assumption made that the radioactivity produced in each station of the absorber is entirely emitted. In practice this assumption is effectively true in most shielding problems.

The use of equations (1) and (2) in expressing I_i and R_i in terms of P_i and known quantities is explained:

Equations (1) and (2) may be rewritten in the following manner:

$$I_i = \frac{1}{t_i} I_{i+1} - \frac{r_i}{t_i} R_{i+1} - \frac{1}{t_i} P_i \tag{5}$$

$$R_i = \frac{r_i}{t_i} I_{i+1} + \left(t_i - \frac{r^2{}_i}{t_i}\right) R_{i+1} + \left(1 - \frac{r_i}{t_i}\right) P_i \tag{6}$$

In matrix notation, equations (5) and (6) may be expressed as

$$\begin{bmatrix} I_i \\ \\ R_i \end{bmatrix} = \begin{bmatrix} \frac{1}{t_i} & -\frac{r_i}{t_i} \\ \\ \frac{r_i}{t_i} & t_i - \frac{r^2{}_i}{t_i} \end{bmatrix} \begin{bmatrix} I_{i+1} \\ \\ R_{i+1} \end{bmatrix} + \begin{bmatrix} -\frac{1}{t_i} \\ \\ 1 - \frac{r_i}{t_i} \end{bmatrix} P_i \tag{7}$$

If the square matrix of constants is denoted by $\left[T_i\right]$ and the column matrix of constants by $\left[Q_i\right]$, equation (7) may be simplified to

$$\begin{bmatrix} I_i \\ \\ R_i \end{bmatrix} = \begin{bmatrix} T_i \end{bmatrix} \begin{bmatrix} I_{i+1} \\ \\ R_{i+1} \end{bmatrix} + \begin{bmatrix} Q_i \end{bmatrix} P_i \tag{8}$$

Beginning with the expression for $\begin{bmatrix} I_1 \\ R_1 \end{bmatrix}$, continued substitution of expressions for subsequent $\begin{bmatrix} I_i \\ R_i \end{bmatrix}$ matrices will result in

$$\begin{bmatrix} I_1 \\ \\ R_1 \end{bmatrix} = \left(\prod_{j=1}^{n} \begin{bmatrix} T_j \end{bmatrix} \right) \begin{bmatrix} I_{n+1} \\ \\ R_{n+1} \end{bmatrix} + \sum_{j=1}^{n} \left(\prod_{k=0}^{j-1} \begin{bmatrix} T_k \end{bmatrix} \right) \begin{bmatrix} Q_j \end{bmatrix} P_j \tag{9}$$

where $\begin{bmatrix} T_0 \end{bmatrix}$ is defined as the identity matrix $\begin{bmatrix} E \end{bmatrix}$.

Similarly, equation (7) can be written as

$$\begin{bmatrix} I_{i+1} \\ \\ R_{i+1} \end{bmatrix} = \begin{bmatrix} t_i - \dfrac{r^2{}_i}{t_i} & \dfrac{r_i}{t_i} \\ \\ -\dfrac{r_i}{t_i} & \dfrac{1}{t_i} \end{bmatrix} \begin{bmatrix} I_i \\ \\ R_i \end{bmatrix} + \begin{bmatrix} 1 - \dfrac{r_i}{t_i} \\ \\ -\dfrac{1}{t_i} \end{bmatrix} P_i \tag{7a}$$

The square matrix of constants is actually $\begin{bmatrix} T_i \end{bmatrix}^{-1}$, whereas the column matrix of constants is $-\begin{bmatrix} T_i \end{bmatrix}^{-1}\begin{bmatrix} Q_i \end{bmatrix}$ so that equation (7a) may be obtained directly from equation (7) by matrix algebra.

If the analogy is extended,

$$\begin{bmatrix} I_{n+1} \\ \\ R_{n+1} \end{bmatrix} = \left(\prod_{j=n}^{1} \begin{bmatrix} T_j \end{bmatrix}^{-1} \right) \begin{bmatrix} I_1 \\ \\ R_1 \end{bmatrix} - \sum_{j=1}^{n} \left(\prod_{k=n}^{j} \begin{bmatrix} T_k \end{bmatrix}^{-1} \right) \begin{bmatrix} Q_j \end{bmatrix} P_j \tag{9a}$$

Although in the following discussion equations of the form of (9a) will be arbitrarily employed to obtain the I_i and R_i in the desired forms, the use of equation (9) leads to the same results.

If matrix notation is temporarily abandoned, equation (9a) is written as two separate equations of the form

$$I_{n+1} = a_1 I_1 + a_2 R_1 + \sum_{j=1}^{n} b_j P_j \tag{10}$$

$$R_{n+1} = a'_1 I_1 + a'_2 R_1 + \sum_{j=1}^{n} b'_j P_j \tag{11}$$

Inasmuch as I_1 and R_{n+1} are constants, equations (10) and (11) may be seen to be in the forms desired. Similar expressions for other I_i and R_i may be found as follows:

The following equation can be written from equation (7a):

$$\begin{bmatrix} I_2 \\ \\ R_2 \end{bmatrix} = \left[T_1\right]^{-1} \begin{bmatrix} I_1 \\ \\ R_1 \end{bmatrix} - \left[T_1\right]^{-1} \left[Q_1\right] P_1 \tag{12}$$

Writing out the two equations represented by equation (12) and substituting equation (11) for R_1 results in expressions for I_2 and R_2 in terms of known quantities and P_i. These expressions may then be used to find I_3 and R_3, which in turn enables the determination of I_4 and R_4 and so forth until all I_i's and R_i's are in the desired forms.

After the expressions for I_i and R_i, which are linear combinations of P_i have been obtained, the next step is to substitute these expressions into forms of equation (4). The resulting set of n differential equations is of the form

$$\left.\begin{aligned}
\frac{dP_1}{d\tau} &= c_1 I_1 + d_1 R_{n+1} + \sum_{j=1}^{n} e_{1,j} P_j \\
\frac{dP_2}{d\tau} &= c_2 I_1 + d_2 R_{n+1} + \sum_{j=1}^{n} e_{2,j} P_j \\
&\vdots \\
\frac{dP_i}{d\tau} &= c_i I_1 + d_i R_{n+1} + \sum_{j=1}^{n} e_{i,j} P_j \\
&\vdots \\
\frac{dP_n}{d\tau} &= c_n I_1 + d_n R_{n+1} + \sum_{j=1}^{n} e_{n,j} P_j
\end{aligned}\right\} \tag{13}$$

where all c's, d's, and e's are arithmetic combinations of t_i, r_i, λ_i, and μ_i, so that all quantities on the right-hand side of equations (13) are constants except the P's.

Equations (13) may be written as the single matrix equation

$$\frac{d[P]}{d\tau} = [B] + [A][P] \tag{14}$$

where $[P]$ is the n-rowed column matrix $\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_i \\ \vdots \\ P_n \end{bmatrix}$

$[A]$ is an n-rowed square matrix, all of whose elements are constants, and $[B]$ is an n-rowed column matrix all of whose elements are linear combinations of I_1 and R_{n+1} and therefore constants.

The solution to equation (14) is obtained in the same manner as if all its terms were scalars.

Thus, if $P = P^0$ at $\tau = \tau_0$, as long as $[A]$ is nonsingular, the solution to equation (14) is

$$[P] = [A]^{-1} \left\{e^{[A](\tau-\tau_0)} \left([A][P^0] + [B]\right) - [B]\right\} \tag{15}$$

(See appendix C for explanation of $e^{[A](\tau-\tau_0)}$.)

where $[P^0]$ is the n-rowed column matrix $\begin{bmatrix} P^0{}_1 \\ P^0{}_2 \\ \cdot \\ \cdot \\ \cdot \\ P^0{}_i \\ \cdot \\ \cdot \\ \cdot \\ P^0{}_n \end{bmatrix}$

After P_i's have been found in terms of known quantities, these P_i's may then be substituted into expressions of the form of equations (10) and (11) to find I_i or R_i. If these expressions are substituted into equation (3), H_i may be found.

Example. - Assume that

(1) The absorber is three stations in thickness.

(2) The incident radiation is monochromatic.

(3) The radiation emitted by the absorber is of the same type and energy as the incident radiation.

(4) $I_1 = 0.1$ roentgen per hour

$R_4 = 0.01$ roentgen per hour

$t_1 = 0.8$

$t_2 = 0.5$

$t_3 = 0.8$

$r_1 = 0.1$

$r_2 = 0.4$

$r_3 = 0.1$

$h_1 = h_2 = h_3 = 5 \times 10^{-2}$

$\mu_1 = \mu_2 = \mu_3 = 5 \times 10^{-7}$ seconds^{-1}

$\lambda_1 = 10^{-5}$ seconds^{-1}

$\lambda_2 = 2 \times 10^{-5}$ seconds^{-1}

$\lambda_3 = 5 \times 10^{-5}$ seconds^{-1}

(5) At $\tau = 0$, $P^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $H_1 = H_2 = H_3 = 0.00$.

The problem is to determine I_4, R_1, and H_2 at $\tau = 10^5$ seconds.

Solution. - If the given data are substituted into equation (12) there results

$$\begin{bmatrix} 0.10 \\ R_1 \end{bmatrix} = \begin{bmatrix} 1.25 & -0.125 \\ .125 & .788 \end{bmatrix} \begin{bmatrix} 2.00 & -0.800 \\ .800 & .180 \end{bmatrix} \begin{bmatrix} 1.25 & -0.125 \\ .125 & .788 \end{bmatrix} \begin{bmatrix} I_4 \\ 0.01 \end{bmatrix}$$

$$+ \begin{bmatrix} -1.25 \\ .875 \end{bmatrix} P_1 + \begin{bmatrix} 1.25 & -0.125 \\ .125 & .788 \end{bmatrix} \begin{bmatrix} -2.00 \\ .200 \end{bmatrix} P_2$$

$$+ \begin{bmatrix} 1.25 & -0.125 \\ .125 & .788 \end{bmatrix} \begin{bmatrix} 2.00 & -0.800 \\ .800 & .180 \end{bmatrix} \begin{bmatrix} -1.25 \\ .875 \end{bmatrix} P_3 \tag{16}$$

Simplifying equation (16) and solving for I_4 and R_1 will result in

$$I_4 = 0.0387 + 0.436\ P_1 + 0.880\ P_2 + 1.356\ P_3 \tag{17}$$

$$R_1 = 0.0420 + 1.356\ P_1 + 0.880\ P_2 + 0.435\ P_3 \tag{18}$$

Similarly,

$$I_2 = 0.0840 + 1.0445\ P_1 + 0.110\ P_2 + 0.0544\ P_3 \tag{19}$$

$$R_2 = 0.040 + 0.445\ P_1 + 1.10\ P_2 + 0.544\ P_3 \tag{20}$$

$$I_3 = 0.047 + 0.544\ P_1 + 1.099\ P_2 + 0.445\ P_3 \tag{21}$$

$$R_3 = 0.0127 + 0.0544\ P_1 + 0.1099\ P_2 + 1.0445\ P_3 \tag{22}$$

From equation (4), the rates of change of P_1, P_2, and P_3 are given by

$$\left.\begin{aligned}
\frac{dP_1}{d\tau} &= 5 \times 10^{-7}\ (0.10 + R_2) - 10^{-5}\ P_1 \\
\frac{dP_2}{d\tau} &= 5 \times 10^{-7}\ (I_2 + R_3) - 2 \times 10^{-5}\ P_2 \\
\frac{dP_3}{d\tau} &= 5 \times 10^{-7}\ (I_3 + 0.01) - 5 \times 10^{-5}\ P_3
\end{aligned}\right\} \qquad (23)$$

where I_2, R_2, I_3, and R_3 are given by equations (19), (20), (21), and (22), respectively.

Substituting these expressions into equation (23) and converting to matrix notation results in

$$\frac{d[P]}{d\tau} = 10^{-6} \begin{bmatrix} -9.778 & 0.549 & 0.272 \\ .549 & -19.890 & .549 \\ .272 & .549 & -49.778 \end{bmatrix} [P] + 10^{-6} \begin{bmatrix} 0.070 \\ .048 \\ .0285 \end{bmatrix} \qquad (24)$$

Applying equation (22) and writing out the equations for each P_i results in

$$\left.\begin{aligned}
P_1 &= -\ 7.42\times10^{-3}\ e^{\xi_1\tau} + 1.22\times10^{-4}\ e^{\xi_2\tau} + 3.48\times10^{-6}\ e^{\xi_3\tau} + 7.29\times10^{-3} \\
P_2 &= -\ 4.06\times10^{-4}\ e^{\xi_1\tau} - 2.24\times10^{-3}\ e^{\xi_2\tau} + 9.87\times10^{-6}\ e^{\xi_3\tau} + 2.63\times10^{-3} \\
P_3 &= -\ 5.62\times10^{-5}\ e^{\xi_1\tau} - 4.00\times10^{-5}\ e^{\xi_2\tau} - 5.46\times10^{-4}\ e^{\xi_3\tau} + 6.42\times10^{-4}
\end{aligned}\right\}$$

(25)

where

$\xi_1 = -\ 9.746 \times 10^{-6}$

$\xi_2 = -\ 19.910 \times 10^{-6}$

$\xi_3 = -\ 49.790 \times 10^{-6}$

Substituting equation (25) into equations (17) and (18) will give I_4 and R_1. Thus at $\tau = 10^5$ seconds,

$$I_4 = 0.0434 \text{ roentgen per hour} \tag{26}$$

$$R_1 = 0.0527 \text{ roentgen per hour} \tag{27}$$

By the use of equation (25), first I_2 and R_3 and then H_2 may be found. If the given data and the newly found expressions for I_2 and R_3 are substituted into equation (3),

$$\begin{aligned} H_2 &= h_2 \left(I_2 + R_3\right) \\ &= 5 \times 10^{-2} \left(0.0850 + 0.0138\right) \\ &= 4.94 \times 10^{-4} \text{ roentgen per hour} \end{aligned}$$

Case II

The assumptions used in case II are:

(1) Stations of absorber are parallel plane surfaces.

(2) Radiation is polychromatic and normal to absorber.

(3) Energy is degraded.

(4) Part of energy absorbed is transformed to energy of radioactivity and part to thermal energy.

(5) Absorber emits radiation of same type as incident radiation.

(6) Radioactivity produced in each station of absorber is entirely emitted.

Method. - In order to make the method of case I amenable to an analysis of polychromatic radiation, the terms are defined as follows:

The energy spectrum of the radiation, which includes the spectrum of the radioactivity in the absorber, is divided into a finite number m of energy bands. Each band is assigned a single energy, which represents all energies within the band; for example, if the limits of a band are 0.50 million electron volt (Mev) and 1.49 Mev, and if 1.00 Mev represents all energies in the band, then energies such as 0.51, 0.70, and 1.20 Mev would all be treated as energies of exactly 1.00 Mev. By varying the number of bands, any desired degree of accuracy may be attained.

From the preceding paragraph, $I_i = \sum_{j=1}^{n} i_{i,j}$ where $i_{i,j}$ is the power of radiation within the j^{th} energy band incident upon the i^{th} station of the absorber. Each I_i may also be expressed in the form of an m-rowed column matrix

$$[I_i] = \begin{bmatrix} i_{i,1} \\ \cdot \\ \cdot \\ \cdot \\ i_{i,j} \\ \cdot \\ \cdot \\ \cdot \\ i_{i,m} \end{bmatrix}$$

Similarly, R_i and P_i may be expressed as the m-rowed column matrices $\begin{bmatrix} \rho_{i,1} \\ \cdot \\ \cdot \\ \cdot \\ \rho_{i,j} \\ \cdot \\ \cdot \\ \cdot \\ \rho_{i,m} \end{bmatrix}$ and $\begin{bmatrix} p_{i,1} \\ \cdot \\ \cdot \\ \cdot \\ p_{i,j} \\ \cdot \\ \cdot \\ \cdot \\ p_{i,m} \end{bmatrix}$, respectively.

With these considerations, relations analogous to equations (1) through (4) may be defined as follows:

$$[I_{i+1}] = [t_i][I_i] + [r_i][R_{i+1}] + [P_i] \tag{28}$$

$$[R_i] = [r_i][I_i] + [t_i][R_{i+1}] + [P_i] \tag{29}$$

$$\frac{d[P_i]}{d\tau} = [\mu_i][I_i + R_{i+1}] - [\lambda_i][P_i] \tag{30}$$

$$[H_i] = [h_i][I_i + R_{i+1}] \tag{31}$$

where $[t_i]$, $[r_i]$, $[\mu_i]$, $[\lambda_i]$, and $[h_i]$ are m^{th} order square matrices, whose elements $t_{j,k}$, $r_{j,k}$, $\mu_{j,k}$, $\lambda_{k,k}$, and $h_{j,j}$, respectively, denote (1) the fraction of incident radiation in energy band k transmitted to energy band j; (2) the fraction of incident radiation in energy band k back-reflected to energy band j; (3) the rate of absorption of radiation power from energy band k to become power of radioactivity in energy band j; (4) the decay constants of radioactivity in the k^{th} energy band; and (5) the rate at which nuclear radiation in the energy band j is transformed to thermal energy.

Before continuing the analysis, the forms of the aforementioned matrices may be simplified. For this purpose, the convention will be adopted of dividing the energy spectrum into bands of increasing radiation energy so that the band of lowest radiation energy will be the first band while the m^{th} band will be the highest radiation energy in the spectrum. By so doing, $[t_i]$ and $[r_i]$ can be reduced to triangular matrices. Inasmuch as transfer of radiation from any energy band to a higher one can occur only after a nuclear reaction, and because the results of all nuclear reactions will appear as p_i or H_i, there will be no elements of $[t_i]$ or $[r_i]$ denoting an upgrading of energy.

This observation is general and applies to any type of nuclear radiation. However, further reduction of $[t_i]$ or $[r_i]$ to diagonal matrices is prevented if allowances are to be made for loss of energy due to either elastic or Compton back-scattering.

No great simplification can be made for $[\mu_i]$. However, at present, there appears to be no experimental evidence of any radiation that excites a radioactivity of both the same type and energy as itself. This fact would serve to put zeros in the diagonal elements of $[\mu_i]$.

By definition, $[\lambda_i]$ and $[h_i]$ are diagonal matrices.

In order to continue the analysis, equations (28) and (29) are solved for I_i and R_i in terms of I_{i+1}:

$$[I_i] = [t_i]^{-1}[I_{i+1}] - [t_i]^{-1}[r_i][R_{i+1}] - [t_i]^{-1}[P_i] \tag{32}$$

$$\begin{aligned}[R_i] = [r_i][t_i]^{-1}[I_{i+1}] + ([t_i] - [r_i][t_i]^{-1}[r_i])[R_{i+1}] \\ + ([E] - [r_i][t_i]^{-1})[P_i]\end{aligned} \tag{33}$$

where $[E]$ is defined as the identity matrix of order m.

Inasmuch as all terms in equations (32) and (33) are matrices, the order of multiplication must be carefully preserved. The assumptions made here imply the existence of the inverses of various matrices. Whenever a solution to the physical problem is possible, all matrix inverses as used in this paper will exist.

In matrix notation, equations (32) and (33) may be written

$$\begin{aligned}\begin{bmatrix} I_i \\ R_i \end{bmatrix} &= \begin{bmatrix} [t_i]^{-1} & -[t_i]^{-1}[r_i] \\ [r_i][t_i]^{-1} & [t_i] - [r_i][t_i]^{-1}[r_i] \end{bmatrix} \begin{bmatrix} I_{i+1} \\ R_{i+1} \end{bmatrix} \\ &+ \begin{bmatrix} -[t_i]^{-1} \\ [E] - [r_i][t_i]^{-1} \end{bmatrix} P_i\end{aligned} \tag{34}$$

If the notation is simplified,

$$\begin{bmatrix} I_i \\ R_i \end{bmatrix} = [T_i] \begin{bmatrix} I_{i+1} \\ R_{i+1} \end{bmatrix} + [Q_i][P_i] \tag{35}$$

where $[T_i]$ is a 2m-rowed square matrix and $[Q_i]$ is a 2m-by-m rectangular matrix.

Beginning with the expression for $\begin{bmatrix} I_1 \\ R_1 \end{bmatrix}$, continued substitution of subsequent I_i and R_i yields

$$\begin{bmatrix} I_1 \\ \\ R_1 \end{bmatrix} = \left(\prod_{j=1}^{n}[T_j]\right)\begin{bmatrix} I_{n+1} \\ \\ R_{n+1} \end{bmatrix} + \sum_{j=1}^{n}\left(\prod_{k=0}^{j-1}[T_k]\right)[Q_j][P_j] \tag{36}$$

Both j and k must be taken in the order indicated.

There are $2m$ equations in $2m$ unknowns so that the i_{n+1} and the ρ_1 may be found in terms of the $p_{i,j}$ and known quantities.

From this point on, the method is the same as the procedure in case I beginning with equation (12).

Example. - Assume that:

(1) The absorber is three stations in thickness.

(2) The incident radiation is composed of two monochromatic radiations of equal intensity.

(3) There is no back-scattering to the absorber.

(4) The radiation emitted by the absorber is of the same type, but of a different energy, than the incident radiation.

(5)

$$I_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$R_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(P^0{}_i)_{\tau_0=0} = (H_i)_{\tau_0=0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[t_1] = \begin{bmatrix} 0.50 & 0.00 & 0.00 \\ .01 & .60 & .00 \\ .01 & .01 & .50 \end{bmatrix}$$

$$[t_2] = \begin{bmatrix} 0.60 & 0.00 & 0.00 \\ .01 & .50 & .00 \\ .01 & .01 & .50 \end{bmatrix}$$

$$[t_3] = \begin{bmatrix} 0.50 & 0.00 & 0.00 \\ .01 & .50 & .00 \\ .01 & .01 & .60 \end{bmatrix}$$

$$[r_1] = \begin{bmatrix} 0.40 & 0.00 & 0.00 \\ .01 & .30 & .00 \\ .01 & .01 & .30 \end{bmatrix}$$

$$[r_2] = \begin{bmatrix} 0.30 & 0.00 & 0.00 \\ .01 & .40 & .00 \\ .01 & .01 & .40 \end{bmatrix}$$

$$[r_3] = \begin{bmatrix} 0.40 & 0.00 & 0.00 \\ .01 & .40 & .00 \\ .01 & .01 & .30 \end{bmatrix}$$

$$[\mu_1] = 10^{-2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{second}^{-1}$$

$$[\mu_2] = 10^{-2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{second}^{-1}$$

$$[\mu_3] = 10^{-2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{second}^{-1}$$

$$[\lambda_1] = [\lambda_2] = [\lambda_3] = 10^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{second}^{-1}$$

$$[h_1] = [h_2] = [h_3] = 10^{-2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The problem is to determine I_4 and H_1 at $\tau = 1000$ seconds.

If the above data are substituted into equation (36),

$$\begin{bmatrix} I_1 \\ R_1 \end{bmatrix} = [T_1][T_2][T_3] \begin{bmatrix} I_4 \\ R_4 \end{bmatrix} + [Q_1][P_1] + [T_1][Q_2][P_2]$$

$$+ [T_1][T_2][Q_3][P_3] \tag{37}$$

where

$$[T_1] = \begin{bmatrix} 2.000 & 0.000 & 0.000 & -0.800 & 0.000 & 0.000 \\ -.033 & 1.67 & .000 & -.00335 & -.500 & .000 \\ -.039 & -.033 & 2.000 & -.00395 & -.010 & -.600 \\ .800 & .000 & .000 & .180 & .000 & .000 \\ .01 & .500 & .000 & .001 & .450 & .000 \\ .00788 & .00668 & .600 & .00078 & .002 & .320 \end{bmatrix}$$

$$[Q_1] = \begin{bmatrix} -2.000 & 0.000 & 0.000 \\ .033 & -1.67 & .000 \\ .039 & .033 & -2.000 \\ .200 & .000 & .000 \\ -.010 & .500 & .000 \\ -.00788 & -.00668 & .400 \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} -1.67 & 0.000 & 0.000 & -0.500 & 0.000 & 0.000 \\ -.033 & 2.000 & .000 & -.010 & -.800 & .000 \\ -.033 & .040 & 2.000 & -.010 & -.036 & -.800 \\ .500 & .000 & .000 & .450 & .000 & .000 \\ .00335 & .800 & .000 & .001 & .180 & .000 \\ .00326 & .004 & .800 & .001 & -.012 & .180 \end{bmatrix}$$

$$[Q_2] = \begin{bmatrix} -1.67 & 0.000 & 0.000 \\ .033 & -2.000 & .000 \\ .033 & -.040 & -2.000 \\ .500 & .000 & .000 \\ -.00335 & .200 & .000 \\ -.00326 & -.004 & .200 \end{bmatrix}$$

$$[T_3] = \begin{bmatrix} 2.000 & 0.000 & 0.000 & -0.800 & 0.000 & 0.000 \\ -.040 & 2.000 & .000 & -.004 & -.800 & .000 \\ -.0333 & -0.0333 & 1.667 & -.00326 & -.00335 & -.500 \\ .800 & .000 & .000 & .180 & .000 & .000 \\ .004 & .800 & .000 & .0004 & .180 & .000 \\ .00979 & .010 & .500 & .001 & .001 & .450 \end{bmatrix}$$

$$[Q_3] = \begin{bmatrix} -2.000 & 0.000 & 0.000 \\ .040 & -2.000 & .000 \\ .033 & .033 & -1.67 \\ .200 & .000 & .000 \\ -.004 & .200 & .000 \\ -.00979 & -.010 & .500 \end{bmatrix}$$

If the indicated operations are performed,

$$\begin{bmatrix} I_1 \\ R_1 \end{bmatrix} = \begin{bmatrix} 4.79 & 0 & 0 & -2.60 & 0 & 0 \\ .353 & 4.73 & 0 & .079 & -2.60 & 0 \\ -.390 & -.430 & 4.668 & .0929 & .114 & -2.60 \\ 2.590 & 0 & 0 & -1.20 & 0 & 0 \\ -.0589 & 2.46 & 0 & -.009 & -1.15 & 0 \\ -.0755 & .0806 & 2.23 & -.0022 & .0002 & -1.03 \end{bmatrix} \begin{bmatrix} I_4 \\ R_4 \end{bmatrix}$$

$$+ \begin{bmatrix} -2 & 0 & 0 \\ .033 & -1.67 & 0 \\ .039 & .033 & -2 \\ .2 & 0 & 0 \\ -.01 & .5 & 0 \\ -.00788 & -.00688 & .4 \end{bmatrix} [P_1] + \begin{bmatrix} -3.73 & 0 & 0 \\ .111 & -3.43 & 0 \\ .128 & .145 & -4.00 \\ -1.24 & 0 & 0 \\ -.001 & -.91 & 0 \\ .0071 & -.011 & -1.24 \end{bmatrix} [P_2]$$

$$+ \begin{bmatrix} -6.14 & 0 & 0 \\ .351 & -6.15 & 0 \\ .388 & .449 & -6.67 \\ -2.91 & 0 & 0 \\ .049 & -2.78 & 0 \\ .067 & .073 & -2.83 \end{bmatrix} [P_3] \tag{38}$$

Substituting the values for I_1 and R_4 into equation (38) and solving for I_4 and R_1 yields:

$$\left.\begin{aligned}
i_{41} &= 0.209 + 0.418p_{11} + 0.781p_{21} + 1.285p_{31} \\
i_{42} &= 0.227 + 0.0242p_{11} + 0.353p_{12} + 0.0348p_{21} + 0.726p_{22} + 0.0217p_{31} \\
&\qquad + 1.301p_{32} \\
i_{43} &= 0.0384 + 0.0287p_{11} + 0.0254p_{12} + 0.428p_{13} + 0.0410p_{21} + 0.0358p_{22} \\
&\qquad + 0.857p_{23} + 0.0262p_{31} + 0.0241p_{32} + 1.428p_{33} \\
\rho_{11} &= 0.542 + 1.284p_{11} + 0.781p_{21} + 0.420p_{31} \\
\rho_{12} &= 0.547 + 0.0250p_{11} + 1.370p_{12} + 0.0388p_{21} + 0.880p_{22} \\
&\qquad + 0.0276p_{31} + 0.423p_{32} \\
\rho_{13} &= 0.0517 + 0.0227p_{11} + 0.0216p_{12} + 1.356p_{13} + 0.0369p_{21} + 0.0325p_{22} \\
&\qquad + 0.715p_{23} + 0.0258p_{31} + 0.0222p_{32} + 0.356p_{33}
\end{aligned}\right\} \qquad (39)$$

As in case I, R_2, I_2, R_3, and I_3 are found in terms of $[I_4]$ and $[R_1]$. All these expressions are then substituted into equation (30) so that the matrix $[A]$ of equation (15) becomes

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-.09366 & -.09352 & -.09288 & .01642 & .01503 & .01430 & .00906 & .00735 & .00712 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
.01596 & .000316 & .01428 & -.08693 & -.09949 & -.09142 & .01599 & .000312 & .01428 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
.000651 & .00737 & .00713 & .000800 & .01488 & .01429 & -.09934 & -.09368 & -.09287
\end{bmatrix}$$

whereas

$$[B] = \begin{bmatrix} 0 \\ 0 \\ .02724 \\ 0 \\ 0 \\ .00846 \\ 0 \\ 0 \\ .00496 \end{bmatrix}$$

The solution to equation (15) is thus:

$$\left.\begin{aligned}
p_{11} &= p_{12} = p_{21} = p_{22} = p_{31} = p_{32} = 0 \\
p_{13} &= -0.1115e^{-0.1000\tau} - 0.03639e^{-0.1090\tau} \\
&\quad - 0.1779e^{-0.06818\tau} + 0.3257 \\
p_{23} &= 0.0000468e^{-0.1000\tau} + 0.05914e^{-0.1090\tau} \\
&\quad - 0.2187e^{-0.06818\tau} + 0.1595 \\
p_{33} &= 0.1113e^{-0.1000\tau} - 0.03626e^{-0.1090\tau} \\
&\quad - 0.1779e^{-0.06818\tau} + 0.1030
\end{aligned}\right\} \tag{40}$$

If these expressions are substituted into equation (39), at $\tau = 1000$ seconds

$$i_{41} = 0.209 \text{ roentgen per hour}$$

$$i_{42} = 0.227 \text{ roentgen per hour}$$

$$i_{43} = 0.329 \text{ roentgen per hour}$$

After R_2 has been found in a similar manner, the resulting expressions may be substituted into equation (31) from which H_1 is found to be $H_1 = 0.0326$ roentgen per hour.

Case III

The assumptions used in case III are:

(1) Radiation consists of polychromatic neutron and gamma radiation.

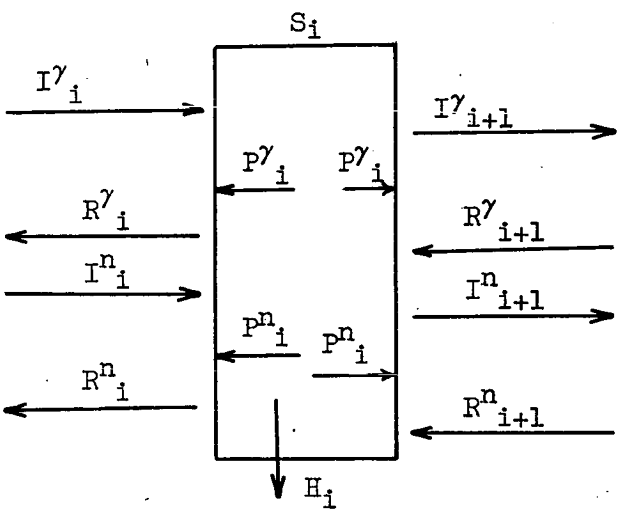
(2) Radiation is normal to absorber.

(3) Part of energy absorbed is transformed to energy of radioactivity and part to thermal energy.

(4) Absorber emits radiations of same types as incident radiation.

(5) Radioactivity produced in each station of absorber is entirely emitted.

Method. - The method differs little from that suggested in case II. Each type of radiation is dealt with separately, whenever possible. The energy spectrum of the gamma radiation is divided into m energy bands and that of the neutron radiation into q energy bands. These energy bands have the same denotations as in case II. In the analysis that follows, symbols for gamma-radiation power will be signified by the superscript γ and those for neutron-radiation power by the superscript n. As may be readily seen from the following description, the method is immediately applicable to any number of different types of radiation.



The following relations hold for this case:

$$[I^{\gamma}{}_{i+1}] = [t^{\gamma}{}_{i}][I^{\gamma}{}_{i}] + [r^{\gamma}{}_{i}][R^{\gamma}{}_{i+1}] + [P^{\gamma}{}_{i}] \tag{41}$$

$$[R^{\gamma}{}_{i}] = [r^{\gamma}{}_{i}][I^{\gamma}{}_{i}] + [t^{\gamma}{}_{i}][R^{\gamma}{}_{i+1}] + [P^{\gamma}{}_{i}] \tag{42}$$

$$\frac{d[P^{\gamma}{}_{i}]}{d\tau} = [\mu^{\gamma}{}_{i}][I^{\gamma}{}_{i} + R^{\gamma}{}_{i+1}] + [\mu^{n,\gamma}{}_{i}][I^{n}{}_{i} + R^{n}{}_{i+1}] - [\lambda^{\gamma}{}_{i}][P^{\gamma}{}_{i}] \tag{43}$$

$$[I^n_{i+1}] = [t^n_i][I^n_i] + [r^n_i][R^n_{i+1}] + [P^n_i] \tag{44}$$

$$[R^n_i] = [r^n_i][I^n_i] + [t^n_i][R^n_{i+1}] + [P^n_i] \tag{45}$$

$$\frac{d[P^n_i]}{d\tau} = [\mu^n_i][I^n_i + R^n_{i+1}] + [\mu^{\gamma,n}_i][I^\gamma_i + R^\gamma_{i+1}] - [\lambda^n_i][P^n_i] \tag{46}$$

$$[H_i] = [h^\gamma_i][I^\gamma_i + R^\gamma_{i+1}] + [h^n_i][I^n_i + R^n_{i+1}] \tag{47}$$

where $[t^\gamma_i]$, $[r^\gamma_i]$, $[\mu^\gamma_i]$, $[\lambda^\gamma_i]$, and $[h^\gamma_i]$ have the same meanings and are in fact identical with $[t_i]$, $[r_i]$, $[\mu_i]$, $[\lambda_i]$, and $[h_i]$, respectively, in case II; the corresponding constants for neutron radiation have analogous meanings: $[\mu^{n,\gamma}_i]$ is an m-by-q rectangular matrix whose elements $\mu_{n_j\gamma_k}$ denote the rate at which neutron radiation of energy j is converted to radioactive power of gamma radiation of energy k, and correspondingly, $[\mu^{\gamma,n}_i]$ is a q-by-m rectangular matrix whose elements $\mu_{\gamma_u n_v}$ denote the rate at which gamma radiation of energy u is converted to radioactive power of neutron radiation of energy v.

The set of equations (41) and (42) is solved to obtain all $[I^\gamma_i]$ and $[R^\gamma_i]$ in terms of known quantities and $[P^\gamma]$ by the same procedure as of case II. Then equations (44) and (45) are treated similarly so that $[I^n_i]$ and $[R^n_i]$ are obtained as expressions consisting of linear combinations of known quantities and $[P^n_i]$.

The resulting expressions are substituted into the set of equations (43) and (46), which is combined to form the single matrix equation

$$\frac{d[P]}{d\tau} = [A][P] + [B]$$

where now [P] is the n(m+q)-rowed column matrix

$$\begin{bmatrix} P^{\gamma}{}_{1} \\ \cdot \\ \cdot \\ \cdot \\ P^{\gamma}{}_{m} \\ P^{n}{}_{1} \\ \cdot \\ \cdot \\ \cdot \\ P^{n}{}_{q} \end{bmatrix}$$

and [A] and [B] are n(m+q)-rowed square and column matrices of constants, respectively.

After this point is reached, the methods of case I may be applied to complete the solution. Although physical problems might occur where the method of case I will not yield unique results, these cases are very rare.

The preceding method is immediately applicable to an analysis of radiation consisting of any number of different types of radiation for which the required experimental data are available.

DISCUSSION

Advantages of the Method

The advantages of the foregoing method are twofold. First, the inclusion of time-dependent factors greatly increases the number of problems to which the method may be applied. Almost all chemical elements have some radioactive isotopes. Inasmuch as many radioactive substances have short half-lives, this method is a closer approximation to the physical problem than a method that considers the radioactivity to be a time-independent constant.

Another advantage is that all the P_i's may be obtained with approximately the same amount of calculations required to obtain a single P_i without the use of matrices. If matrix methods are not used, the usual procedure is to obtain an n^{th} order equation in one P_i by substitution, solve it by differential operators, and then substitute this solution into the expressions for each of the other P_i's in terms of the one just obtained.

If computing machines are available, the calculations of $[A]^{-1}$ and $e^{[A]\tau}$ do not take long and the matrix method may be readily employed to the fullest advantage.

An example of a possible application of this method is:

Assume that two prospective absorbing materials are available for diminishing the intensity of a certain radiation. One material is an excellent absorber but rapidly becomes radioactive when subjected to the given radiation. The other material, although inferior to the first in its ability to absorb the radiation, lacks the undesirable property of becoming radioactive. Which absorber should be chosen?

Obviously, there are situations in which the second material would be preferable and other situations in which the first material would be preferable. The use of this method could shorten the time spent in making choices of this nature.

Limitations of Method

The main limitation of the method is the amount of available experimental data on the absorption of gamma rays and neutrons and on the rates of conversion of these radiations to secondary radiations and to thermal energy.

CONCLUSION

A matrix method for determining the effectiveness of radioactive absorbers has been set up; theoretically the method is immediately applicable for solution of absorber problems. Adequate experimental data on the absorption of radiation and rates of conversion to secondary radiation and thermal energy are, however, required for utilization of this method.

Lewis Flight Propulsion Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio, February 4, 1949.

APPENDIX A

SYMBOLS

The following symbols are used in this report:

[A], [B]	matrix coefficients
a,a',b,b',c,d,e	scalar coefficients
[E]	identity matrix
H_i	thermal power generated in i^{th} station of absorber
$[H_i]$	matrix of thermal power generated in i^{th} station of absorber
h	rate of conversion of energy of nuclear radiation absorbed to thermal energy
$[h_i]$	matrix of rate of conversion of energy of nuclear radiation absorbed to thermal energy
I_i	power of radiation from $(i-1)^{th}$ station incident on i^{th} station
$[I_i]$	matrix of power of radiation from $(i-1)^{th}$ station incident on i^{th} station
I_{i+1}	power of radiation from i^{th} station incident on $(i+1)^{th}$ station
$[I_{i+1}]$	matrix of power of radiation from i^{th} station incident on $(i+1)^{th}$ station
$i_{i,j}$	power of radiation in j^{th} energy band incident on i^{th} station from $(i+1)^{th}$ station
n	number of stations in absorber
[P]	matrix of power of radiation from radioactivity

$[P^0]$	matrix of initial power of radiation from radioactivity
P_i	one-half of power of radiation from radioactivity in i^{th} station
$p_{i,j}$	matrix element of power of radiation from radioactivity in the i^{th} station
$[Q_i]$	rectangular matrix coefficient
R_i	power of radiation from i^{th} station incident on $(i-1)^{th}$ station
$[R_i]$	matrix of power of radiation from i^{th} station incident on $(i-1)^{th}$ station
R_{i+1}	power of radiation from $(i+1)^{th}$ station incident on i^{th} station
$[R_{i+1}]$	matrix of power of radiation from $(i+1)^{th}$ station incident on i^{th} station
r	power back-scattering coefficient
$[r_i]$	matrix of power back-scattering coefficient
S_i	i^{th} station of absorber
$[T_0]$	identity matrix $[E]$
$[T_i]$	square matrix coefficient
t	power-transmission coefficient
$[t_i]$	matrix of power-transmission coefficient
λ	decay constant for radioactivity, seconds^{-1}
$[\lambda_i]$	matrix of decay constant for radioactivity, seconds^{-1}
μ_i	rate of conversion of energy absorbed in S_i to energy of radioactivity in S_i

$[\mu_i]$	matrix of rate of conversion of energy of radiation absorbed in S_i to energy of radioactivity in S_i
$\prod$	symbol denoting that product of terms following it are to be taken as indicated by indices
$\rho_{i,j}$	power of radiation in j^{th} energy band emerging from i^{th} station
τ	time
τ_0	initial time

Subscripts:

τ	time
τ_0	initial time

Superscripts:

n	neutron radiation
γ	gamma radiation

APPENDIX B

DERIVATION OF EQUATION FOR POWER OF RADIOACTIVITY

IN i^{th} STATION

A station of an absorber S_i is assumed to consist of a very large number of stable atoms and some radioactive atoms and all radioactivity produced in this station is assumed to be emitted. The rate of change of the number of radioactive atoms N_i in the absorber is given by

$$\frac{dN_i}{d\tau} = \frac{a_i\ (I_i + R_{i+1})}{(E_i)_{s,R}} - \lambda_i N_i \tag{B1}$$

where a_i equals the fraction of the radiation energy incident upon the i^{th} station that is absorbed in forming radioactive atoms and $(E_i)_{s,R}$ equals the energy required to change a stable atom to a radioactive one.

Now

$$2P_i = (E_i)_{R,s}\ \lambda_i N_i \tag{B2}$$

where $(E_i)_{R,s}$ is the energy emitted when a radioactive atom changes to a stable one. If equation (B2) is differentiated with respect to time,

$$2\ \frac{dP_i}{d\tau} = (E_i)_{R,s}\ \lambda_i\ \frac{dN_i}{d\tau} \tag{B3}$$

Then if equations (B2) and (B3) are substituted into equation (B1) and $\frac{\lambda_i\ (E_i)_{R,s}\ a_i}{2(E_i)_{s,R}}$ is set equal to μ_i,

$$\frac{dP_i}{d\tau} = \mu_i\ (I_i + R_{i+1}) - \lambda_i P_i \tag{B4}$$

APPENDIX C

DEFINITION OF "e" TO A MATRIX POWER

Inasmuch as [A] is a matrix, $e^{[A]\tau}$ must be defined. The symbol $e^{[A]\tau}$ is defined as $\sum_{j=0}^{\infty} \frac{([A]\tau)^j}{j!}$, which may be evaluated more readily with the aid of the confluent form of Sylvester's Theorem (reference 3).

Let [u] be any square matrix of rank q with r distinct characteristic roots $\lambda_1 \ldots \lambda_r$, of multiplicity $\alpha_1 \ldots \alpha_r$, respectively. Then if P([u]) is a polynominal in [u] or a convergent infinite series in [u], the theorem states that

$$P([u]) = \sum_{i=1}^{r} \left\{ \frac{1}{(\alpha_i - 1)!} \frac{d^{\alpha_i - 1}}{d\lambda^{\alpha_i - 1}} \left[\frac{P(\lambda) \prod_{k \neq i} ([u] - \lambda_k [E])}{\prod_{k \neq i} (\lambda_i - \lambda_k)^{\alpha_i}} \right]_{\lambda = \lambda_i} \right\} \tag{C1}$$

where [E] is the identity matrix of order q.

If all the roots are distinct then equation (C1) reduces to

$$P([u]) = \sum_{i=1}^{q} \left[\frac{P(\lambda_i) \prod_{k \neq i} ([u] - \lambda_k [E])}{\prod_{k \neq i} (\lambda_i - \lambda_k)} \right]$$

<u>Example 1.</u> - Calculate $e^{[u]}$ when $[u] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 3 \end{bmatrix}$. Then the characteristic roots are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, and

$$e^{[u]} = e^{1}\left[\frac{(2[E] - [u])(3[E] - [u])}{(2 - 1)(3 - 1)}\right] + e^{2}\left[\frac{([E] - [u])(3[E] - [u])}{(1 - 2)(3 - 2)}\right]$$

$$+ e^{3}\left[\frac{([E] - [u])(2[E] - [u])}{(1 - 3)(2 - 3)}\right]$$

from which

$$e^{[u]} = e\begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 2.5 & 0 & 0 \end{bmatrix} + e^{2}\begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & 0 \\ 8 & 4 & 0 \end{bmatrix} + e^{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 5.5 & 4 & 1 \end{bmatrix}$$

<u>Example 2.</u> - Calculate $e^{[u]}$ where $[u] = \begin{bmatrix} 2 & -2 & 3 \\ 10 & -4 & 5 \\ 5 & -4 & 6 \end{bmatrix}$

Here $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$ so that

$$e^{[u]} = \left[\frac{d}{d\lambda}\,\frac{e^{\lambda}([u] - \lambda[E])}{\lambda - 2}\right]_{\lambda=1} + \frac{e^{2}([u] - 2[E])}{(2 - 1)^{2}}$$

$$= e\begin{bmatrix} 5 & 4 & -8 \\ 15 & 16 & -30 \\ 10 & 10 & -19 \end{bmatrix} + e\begin{bmatrix} 5 & 2 & -5 \\ 25 & 10 & -25 \\ 15 & 6 & -15 \end{bmatrix} + e^{2}\begin{bmatrix} -4 & -4 & 8 \\ -15 & -15 & 30 \\ -10 & -10 & 20 \end{bmatrix}$$

$$= e\begin{bmatrix} 10 & 6 & -13 \\ 40 & 26 & -55 \\ 25 & 16 & -34 \end{bmatrix} + e^{2}\begin{bmatrix} -4 & -4 & 8 \\ -15 & -15 & 30 \\ -10 & -10 & 20 \end{bmatrix}$$

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